Testing a Multivariate Model for Wavelet Coefficients

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Motivation

Our Model: **Multivariate Power Exponential (MEP)** distribution

- Previously used in image retrieval [Verdoolaege et al., 2009] or image watermarking [Kwitt et al., 2009]
- Multivariate extension to the Generalized Gaussian distribution (GGD)
- We miss a reasonable statistical tool to check the Goodness-of-Fit

Research Objective

Check the assumption

$$\underbrace{\boldsymbol{X}_1, \ldots, \boldsymbol{X}_N}_{\text{Sample}} \sim \text{MEP}, \quad \boldsymbol{X}_i \in \mathbb{R}^n$$

in terms of hypothesis testing, i.e. "is there any evidence against?"

The MEP Distribution

The MEP belongs to the family of elliptical distributions

Probability Density Function (pdf)

$$p(\boldsymbol{x}; \boldsymbol{\mu}, \boldsymbol{\Sigma}, \beta) = \frac{n\Gamma\left(\frac{n}{2}\right)}{\pi^{n/2}\Gamma\left(1 + \frac{n}{2\beta}\right) 2^{1+\frac{n}{2\beta}}} |\boldsymbol{\Sigma}|^{-1/2}$$

$$\exp\left\{-\frac{1}{2}\left[(\boldsymbol{x} - \boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1}(\boldsymbol{x} - \boldsymbol{\mu})\right]^{\beta}\right\}, \boldsymbol{x} \in \mathbb{R}^n$$

$$\underbrace{\boldsymbol{\mu} \in \mathbb{R}^n}$$

Location (assumed **0** for wavelet coefficients [Mallat, 1999])

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$$\underbrace{\boldsymbol{\Sigma} \in \mathbb{M}_d}_{\text{Positive-definite (symmetric) dispersion matrix}}$$

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$$\underbrace{\beta \in \mathbb{R}_+}_{\text{Shape}}$$

The MEP Distribution

The MEP admits the following stochastic representation ...

Stochastic Representation

$$\boldsymbol{X} \sim R\boldsymbol{A}^T\boldsymbol{r}^{(n)}$$

with

$$\boldsymbol{\Sigma} = \boldsymbol{A}^T\boldsymbol{A}$$

$$\underbrace{\boldsymbol{r}^{(n)} \sim \mathbb{S}^{n-1}}_{\text{(uniformly distributed on unit-sphere in } \mathbb{R}^n\text{)}}$$

$$R \sim F_R, \quad p_R(r;\beta) = \frac{n}{\Gamma\left(1+\frac{n}{2\beta}\right)2^{\frac{n}{2\beta}}} r^{n-1} \exp\left\{-\frac{1}{2}r^{2\beta}\right\} \mathbf{1}_{(0,\infty)}(r)$$

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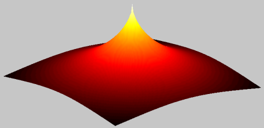
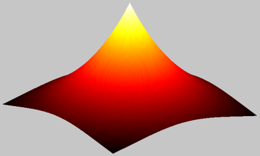
Examples (in $\mathbb{R}^3$)

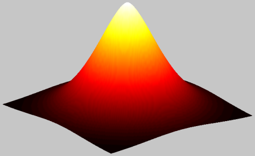
In all examples: $\boldsymbol{\Sigma} = \boldsymbol{I}$



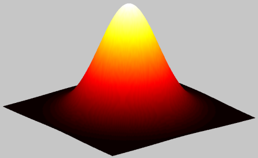
(a) $\beta = 0.3$



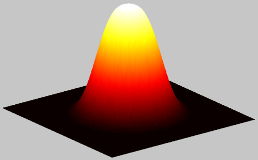
(b) $\beta = 0.5$



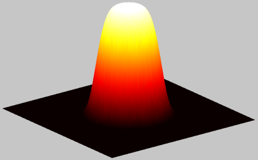
(c) $\beta = 0.8$



(d) $\beta = 1.0$ (Gaussian)



(e) $\beta = 1.5$



(f) $\beta = 2.5$

Elements of the Proposed GoF Test

Covered in this talk

- Computation of a suitable Test Statistic
- Test strategy (following the work of [Smith and Jain, 1988])
- Sampling from the MEP distribution

Not covered in this talk (see paper)

- Parameter estimation (Maximum-Likelihood, Method of Moments, etc.)

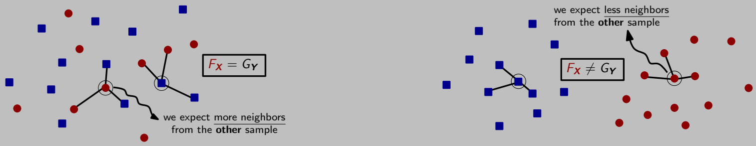
The Test Statistic

Our approach is motivated by an idea from the field of Two-Sample Hypothesis Testing [Schilling, 1986]. Given the pooled sample

$$\{\underbrace{\boldsymbol{X}_1, \ldots, \boldsymbol{X}_{N_1}}_{\sim F_{\boldsymbol{X}}}, \underbrace{\boldsymbol{Y}_1, \ldots, \boldsymbol{Y}_{N_2}}_{\sim G_{\boldsymbol{Y}}}\}$$

the objective is to test the hypothesis $F_{\boldsymbol{X}} = G_{\boldsymbol{Y}}$.

The Idea



Statistic

Count the number of k-Nearest Neighbor Coincidences [Schilling, 1986]

Test Strategy

Our approach is an extension to the GoF test by [Smith and Jain, 1988] to test for multivariate normality and yields **two variants**:

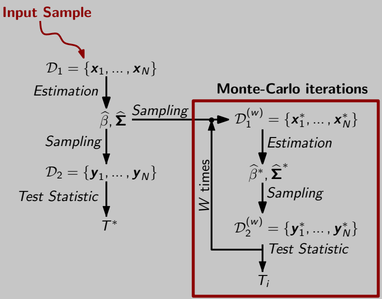
Monte-Carlo Variant

The p-value is estimated based on a resampling strategy

Normal-Approximation Variant

The p-value is estimated based on a normal approximation of our test statistic and a result from [Schilling, 1986]

Test Strategy – Monte-Carlo Variant

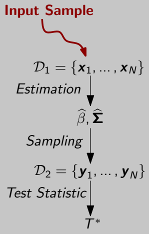


p-value Estimate

Given the above testing procedure, the p-value is estimated by

$$\hat{p} = \frac{\#\{T_i > T^*\} + 0.5}{W + 1} \rightarrow \text{Reject } \mathcal{H}_0 : \boldsymbol{X} \sim \text{MEP in case } \underbrace{\hat{p} < \alpha}_{\text{Test Level } \alpha}$$

Test Strategy – Normal-Approximation Variant



Idea

- Exploit the asymptotic distribution of T^*
- We know from [Schilling, 1986] that

$$\tilde{T} := (Nk)^{1/2} \frac{(T^* - \mu)}{\sigma} \sim \mathcal{N}(0,1)$$

for $N \to \infty$ under $\mathcal{H}_0 : F_{\boldsymbol{X}} = G_{\boldsymbol{Y}}$.

p-value Estimate

Exploiting the fact that $\tilde{T} \sim \mathcal{N}(0,1)$ allows to estimate the *p*-value as

$$\hat{p} = \mathbb{P}(T^* \geq T) = 1 - \underbrace{F_T(\tilde{T})}_{\text{Normal CDF}} \to \text{Reject } \mathcal{H}_0 : \boldsymbol{X} \sim \text{MEP in case } \hat{p} < \alpha$$

Sampling from the MEP Distribution

Strategy: Exploit the stochastic representation $\boldsymbol{X} \sim R\boldsymbol{A}^T\boldsymbol{r}^{(n)}$

Steps to generate $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N$ from MEP with β and $\boldsymbol{\Sigma}$

1. Decompose $\boldsymbol{\Sigma}$ into $\boldsymbol{A}^T\boldsymbol{A}$ using Cholesky decomposition $\rightarrow$ ✓

[a] $P_u(a, x)$ denoting the regularized (upper) incomplete Gamma function

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1. Decompose $\boldsymbol{\Sigma}$ into $\boldsymbol{A}^T\boldsymbol{A}$ using Cholesky decomposition $\rightarrow$ ✓
2. Draw a random sample $r_1, \ldots, r_N$ from F_R as $r_i = F_R^{-1}(u_i, \beta)$ with[a]

$$F_R^{-1}(u;\beta) = 2^{\frac{1}{2\beta}} \left[P_u^{-1}\left({}^{n}\!/_{2\beta}, 1-u\right)\right]^{\frac{1}{2\beta}}, u \in [0,1] \rightarrow ✓$$

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3. Draw a sample $\boldsymbol{r}_1^{(n)}, \ldots, \boldsymbol{r}_N^{(n)}$ from uniform distribution on unit-sphere

$$\boldsymbol{r}_i^{(n)} = (z_1, \ldots, z_N) \quad \text{with} \quad z_i = \frac{y_i}{\sum_i y_i^2}, y_i \sim \mathcal{N}(0,1) \rightarrow \checkmark$$

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4. Use $\forall i : \boldsymbol{x}_i = r_i\boldsymbol{A}^T\boldsymbol{r}_i^{(n)}$ to generate $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N$.

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Evaluation of the Test Size

Protocol:

- Check if the desired significance level α is reached for different N
- Count the number of times $\mathcal{H}_0 : \boldsymbol{X} \sim G_{\boldsymbol{X}}$ is falsely rejected
- Take our implementation of the GoF test in [Gomez et al., 1998] as reference

α	N	Estimated $\hat{\alpha}$		
		Gomez et al. [Gomez et al., 1998]	Monte-Carlo	Normal
0.01	200	0.030	0.002	0.022
	400	0.028	0.001	0.002
	800	0.014	0.001	0.018
0.05	200	0.084	0.022	0.063
	400	0.118	0.012	0.014
	800	0.108	0.053	0.069
0.10	200	0.194	0.044	0.132
	400	0.212	0.026	0.048
	800	0.196	0.084	0.152

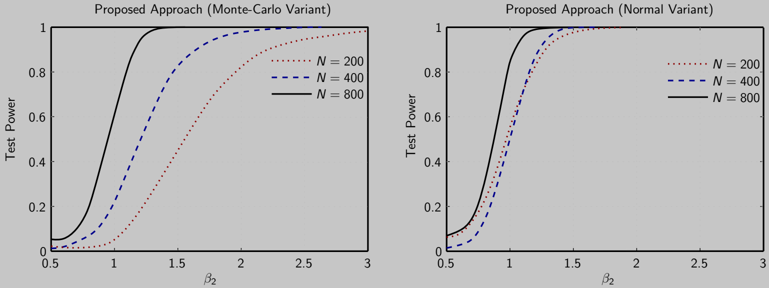
Evaluation of the Test Power

Protocol:

- Take a two-component MEP mixture model

$$p(\boldsymbol{x}, \beta_1, \beta_2, \boldsymbol{\Sigma}_1, \boldsymbol{\Sigma}_2) = \sum_{i=1}^{2} \pi_i p(\boldsymbol{x}, \beta_i, \boldsymbol{\Sigma}_i), \quad \sum_i \pi_i = 1$$

- Set $\beta_2 = \beta_1 + \epsilon_i, \epsilon_i \in \{0.1, 0.2, \ldots, 2.5\}, \boldsymbol{\Sigma}_1 = \boldsymbol{\Sigma}_2 = \boldsymbol{I}$
- $W = 1000$ (for Monte-Carlo variant), perform 1000 trials per ϵ_i



GoF Tests on Real-World Data

- Datasets: Three texture databases, one natural image database
- Determine $\mathcal{H}_0$ rejection rates for all coefficients from a 3-level DWT
- Tests are performed with a significance level of $\alpha = 0.05$
- Rejection rates are averaged over the number of DWT subbands
- Simulate the test of [Smith and Jain, 1988] by fixing $\beta = 1$ (estimate just $\boldsymbol{\Sigma}$)

Model	**Database**			
	Stex[1]	Vistex (full)	Outex	UCID
MEP	25.09	35.13	11.15	56.18
MVN[2] (i.e. $\beta = 1$)	57.13	73.19	39.66	98.97

[1] Salzburg Textures, available at http://www.wavelab.at
[2] Multivariate Normal

Discussion

Summary of this talk:

- Proposed a novel GoF test for the MEP distribution
- Our test includes GoF test of [Smith and Jain, 1988] as a special case
- Regarding the power study, we just tested against shape alternatives

Not in this talk (see paper):

- Proposed an implementation of the GoF test sketched in [Gomez et al., 1998]
- Discussion of estimation issues

Outlook:

- Power study for varying $\boldsymbol{\Sigma}$
- Study impact of estimation on the size/power results (bias ?)

Source Code

Will be available at `http://www.wavelab.at` soon after **ICIP 2011**.

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